where

$$B = \xi_0/\{\eta_0 + \sigma[2/(1+\sigma)]^{1/2}\} \quad (15)$$

and

$$\sigma = R_1/R_a \quad (16)$$

This result, along with the conservation laws and the results of Ref. 4, is sufficient to determine the properties of the transfer orbit uniquely.

Transfer from Inner Orbit to Outer Terminal

The derivation of the solution in this case is exactly analogous to that of the preceding problem. In this case the initial terminal is allowed to vary over the inner orbit and the problem is to find the location of the variable terminal which minimizes the impulse requirement for the optimum terminal-to-terminal transfers.

Since now the initial terminal is variable, the total impulse requirement for the optimum terminal-to-terminal transfer is expressed in dimensionless multiples of circular velocity C_2 at the radial distance of the fixed outer terminal. Equation (1) then becomes

$$\Delta V_T/C_2 = [\{x_F + r[2/(1+r)]^{1/2}\}^2 + y_F^2]^{1/2} - [\{x_0 + [2/(1+r)]^{1/2}\}^2 + y_0^2]^{1/2} \quad (17)$$

where

$$(u_i, v_i) = (C_2x_i, C_2y_i) \quad (18)$$

and

$$r = 1/\rho = R_2/R_1 \quad (19)$$

Denoting the radial distances of the apogee and perigee of the inner orbit, respectively, by R_a' and R_p', the dimensionless perigee velocity x_p on that orbit is given by

$$x_p^2 = 2R_2R_a'/R_p'(R_a' + R_p') \quad (20)$$

As before, x_0, x_F, and x_p have the same sign, which is taken to be positive.

Corresponding to Eqs. (6–8) of the previous case, the conservation laws yield

$$x_0 = kr \quad (21)$$

and

$$y_0^2 = (x_p - 1/k)^2 - (kr - 1/k)^2 \quad (22)$$

where

$$k = R_p'x_p/R_2 = \text{const} \quad (23)$$

With Eqs. (21) and (22), $\Delta V_T/C_2$ is a continuous function of the single variable r in the closed interval

$$R_2/R_p' \geq r \geq R_2/R_a' > 1 \quad (24)$$

Since neither term in Eq. (17) can vanish, $\Delta V_T/C_2$ has a continuous first derivative in the interval of the variable r. Differentiating and simplifying gives

$$\frac{d(\Delta V_T/C_2)}{dr} = \frac{2^{-1/2}(2+r)}{(1+r)^{3/2}}\left\{\frac{x_F + r[2/(1+r)]^{1/2}}{[\{x_F + r[2/(1+r)]^{1/2}\}^2 + y_F^2]^{1/2}} - \frac{k + r[2/(1+r)]^{1/2}}{[\{x_0 + [2/(1+r)]^{1/2}\}^2 + y_0^2]^{1/2}}\right\} \quad (25)$$

The first fraction inside the braces of Eq. (25) is always ≤ 1 so that $\Delta V_T/C_2$ has a negative first derivative when

$$k + r[2/(1+r)]^{1/2} > [\{x_0 + [2/(1+r)]^{1/2}\}^2 + y_0^2]^{1/2} \quad (26)$$

Assuming that the inequality of Eq. (26) does not hold and using Eqs. (20–23) gives

$$1 \geq R_2/R_a' \quad (27)$$

contradicting Eq. (24). Thus $\Delta V_T/C_2$ has a negative first derivative throughout the interval, and the absolute optimum two-impulse transfer from the inner orbit to an arbitrary outer coplanar terminal is a transfer from the perigee of the inner orbit.

From Eq. (1) the total impulse requirement for the optimum transfer is

$$(\Delta V_T)_{\min} = [\{u_F + [2GMR_2/R_p'(R_p' + R_2)]^{1/2}\}^2 + v_F^2]^{1/2} - u_p - [2GMR_p'/R_2(R_p' + R_2)]^{1/2} \quad (28)$$

where u_p is the perigee velocity on the inner orbit.

From the results of Ref. 4 it can be shown that the perigee of the transfer orbit coincides with the perigee of the inner orbit, and that the perigee velocity u_1 on the transfer orbit is

$$u_1 = C_2s\left(\frac{2}{1+s}\right)^{1/2}\left[\frac{1 - s(1+A^2)^{1/2}}{(1+A^2)^{1/2} - s}\right] \quad (29)$$

where

$$A = y_F/\{x_F + s[2/(1+s)]^{1/2}\} \quad (30)$$

and

$$s = R_2/R_p' \quad (31)$$

This result, combined with the conservation laws and the results of Ref. 4, is sufficient to determine the properties of the transfer orbit uniquely.

From the foregoing two results, it follows that the optimum two-impulse transfer between nonintersecting coplanar elliptical orbits is the Hohmann transfer from the inner perigee to the outer apogee, a result already established in Refs. 2 and 3 by a much longer method.

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Step-Temperature Effects on Direct Measurements of Drag

John C. Westkaemper*
University of Texas, Austin, Texas

It is well known that a local discontinuity in the surface temperature of a body moving through a fluid will markedly influence the heat transfer from the fluid to the body region over which the discontinuity exists. Much attention has been given to this step-temperature effect in heat transfer literature. The effect is especially evident in the measurement of heat transfer using plug-type calorimeters, where temperature mismatch is known to cause large errors.[1] The temperature history of the floating element of a direct-measuring skin-friction balance is similar to that of a plug

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* Research Engineer, Defense Research Laboratory. Member AIAA.

Table 1 Typical results for drag balance

Stagnation temp., °R	Plate temp., °R	Disk temp., °R Max.	Disk temp., °R Min.	Drag variation during run, %	
938	638	660	626	+1.6	−0.8
872	627	627	583	+1.2	−0.0
908	595	625	590	+1.3	−0.3
1093	629	685	624	+2.3	−0.1
1004	635	644	594	+0.1	−3.3
875	634	638	585	+0.8	−1.9

Table 2 Typical results for slug calorimeter

Stagnation temp., °R	Plate temp., °R	Disk temp., °R Max.	Disk temp., °R Min.	Drag variation during run, %	
958	578	631	572	+24.7	−46.5
1144	581	643	560	+20.9	−36.6
1048	584	640	564	+28.1	−39.2
958	582	622	559	+28.2	−34.8
957	584	611	560	+38.4	−22.6
1045	583	625	577	+10.6	−29.9

calorimeter, as both are small surface elements that are insulated thermally from the surrounding main surface. The Reynolds analogy that relates skin-friction and heat-transfer coefficients leads one to suspect that the direct-measuring balance might be subject to the same error as found with calorimeters. This note presents the results of an experimental investigation of the effect of temperature mismatch on direct measurements of skin friction, using a floating-element-type balance.

A brief description of the principles of floating-element balances will be given to explain the nature of the problem; further details may be found in Ref. 2. The floating element of this type of balance is usually a disk or rectangular plate mounted flush with the surface of the test body by means of leaf-type cantilever springs extending inside the body. The opening in the body is slightly larger than the element, which thus "floats" free of the body on the leaf springs. The springs are oriented so that they will deflect slightly under the action of the drag force on the element. The force may be determined from the spring constant and deflection under load, or, alternatively, drag is equal to the force required to prevent deflection. The floating element is usually of different thickness from the wall of the test body; hence its heat capacity per unit surface area is also different. Thus, under non-adiabatic flow the element and body temperature normally will differ. The element, therefore, presents a step-temperature discontinuity to the oncoming boundary layer flow unless some type of temperature control system is used to eliminate the discontinuity.

The test body for the experiments reported herein was a flat-plate wind-tunnel model that was maintained at a constant temperature during each run by means of water cooling. The balance floating element was a copper disk having no integral cooling system. Because of aerodynamic heating by the flow, the element temperature tended to rise continuously toward the recovery temperature of the flow. A rudimentary temperature "control" method that gave satisfactory results consisted of ejecting a few drops of water on the disk surface, using an externally mounted, retractable probe.[3] The evaporating water momentarily depressed the disk temperature below the plate temperature, the plate temperature being constant during each run as noted in the foregoing. The probe was retracted rapidly, after which the disk temperature again rose toward the flow recovery temperature. At the instant when the disk temperature equaled the plate temperature, there was no temperature discontinuity between the flat plate and the floating element. Since the element was a disk of high-conductivity copper, and since the heating rates were relatively low, there were no significant temperature gradients within the element itself. It therefore was assumed (and confirmed by comparison with theory) that valid skin friction data for the isothermal plate condition were obtained at this instant of zero temperature difference.

The skin friction force, element temperature, and plate temperature were recorded continuously during each run. These data were used for determining the effect of temperature mismatch by comparing the skin friction under conditions of mismatch with the skin friction obtained at the instant of zero mismatch. The tests were made in a wind tunnel with a nominal Mach number of 5.0, using a flat plate model that completely spanned the 6-in. test section width.[3] The balance floating element was a 1-in.-diam copper disk, mounted with its center 12.6 in. aft of the leading edge of the flat plate. The Reynolds number based on this length was from 5.3 to 8.0×10^6. A total of 35 runs were made covering the most frequently used portion of the tunnel operating envelope.

Table 1 shows the operating conditions and drag variation for six runs that were selected as representative. As previously noted, the drag reference was the value measured when the floating-element and plate temperatures were equal. The element temperature range was from 56°F above the plate temperature to 49°F below. The average drag variation for all 35 runs was approximately $\pm 1\%$; the maximum was -3.3%. There was no apparent correlation of the drag variation with the conditions of temperature mismatch. When the disk temperature was at its maximum value, the corresponding drag variation was positive for approximately half the runs and negative for the remainder. Similar results were obtained when the disk temperature was at a minimum. Thus, the variation in drag did not appear to be related to the mismatch condition, but appeared rather to be random in nature.

The usual repeatability of measurements of drag coefficient with this test system is $\pm 2\%$ for adiabatic flow. Thus, it may be seen that any variable having an effect of less than 2% cannot be determined directly, although a small consistent effect might be distinguished by statistical analysis. Such an analysis was not applied to these data, since not even a gross correlation was apparent. It is believed that the variation obtained was probably the usual random variation obtained with this test equipment.

It is interesting to compare the foregoing results with similar results obtained from a copper slug calorimeter, also of 1-in. diam. The data of Ref. 3 were analyzed for this purpose, and representative results are shown in Table 2. It is evident that a very strong effect of temperature mismatch exists when measuring heat transfer. The heat transfer test conditions were very similar to those of Table 1; thus it is apparent that the usual Reynolds analogy between heat transfer and skin friction is not valid when a surface temperature discontinuity exists. This result is not surprising when one considers the parameters that govern heat transfer and skin friction. Heat transfer is determined by the conductivity and temperature profile of the fluid at the wall; skin friction is expressed similarly in terms of viscosity and velocity profile, also at the wall. Therefore, it is to be expected that a temperature discontinuity would influence heat transfer and skin friction to different degrees. The absence of a measurable influence on skin friction was not anticipated, however. Since the temperature discontinuity in these experiments extended over only a short streamwise length (1 in.), its effect on the viscosity and velocity profiles was either very small or opposite in direction and thus self-canceling.

It is concluded that, for the test conditions covered in this work, the effect, if any, of moderate temperature mismatch between the element of a floating-element balance and the surrounding surface is less than 2% of the drag obtained when there is no mismatch. Although the test equipment was not capable of resolving an effect smaller than 2%, the nature of the results strongly indicated that the effect was even smaller.

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COMMENTS

Comment on "Satellite Motions About an Oblate Planet"

J. Kevorkian* and J. F. Murphy†
California Institute of Technology, Pasadena, Calif.

THE solution presented in Ref. 1 is valid only for a time interval of order unity since θ_1 contains the mixed secular term

$$\frac{-R^2 \sin 2i}{2c^4 r_0^2}\,\xi \cos\xi$$

In this note it will be shown that this secular term can be eliminated by expressing the solution in a frame that rotates at an appropriate uniform rate about the polar axis. This technique was used[2] for satellite motions in the restricted three-body problem.

Consider the coordinate system x^*, y^*, z^* defined with respect to the X, Y, Z system of Ref. 1 by

$$\begin{aligned} x^* &= X\cos\Omega + Y\sin\Omega \\ y^* &= -X\cos i\sin\Omega + Y\cos i\cos\Omega + Z\sin i \\ z^* &= X\sin i\sin\Omega - Y\sin i\cos\Omega + Z\cos i \end{aligned} \tag{1}$$

In Eqs. (1), Ω is the longitude of the node that is assumed to rotate slowly in the X-Y plane [cf., discussion below Eq. (10)].

If the analysis of Ref. 1 is paralleled using the starred variables of Eqs. (1) instead of the X, Y, Z variables, one can derive the three equations corresponding to (3.15) of Ref. 1. In the present context r, θ, φ, u, and P are all formally related to starred variables by the same equations used in Ref. 1. Since Ω now depends on time, the following additional terms will appear on the left-hand sides of Eqs. (3.15):

Equation for u

$$(2\cos i\sin^2\theta + 2\cos\varphi\sin i(d\theta/d\varphi) - \sin 2\theta\sin\varphi\sin i)Pu(d\Omega/d\varphi) + O(J^2) \tag{2}$$

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* Research Fellow in Aeronautics; also Consultant, Douglas Aircraft Company Inc.

† Graduate Student in Aeronautics; also Aerodynamicist, Douglas Aircraft Co. Inc. (June–September 1962).

Equation for θ

$$(d/d\varphi)[P(d\Omega/d\varphi)\cos\varphi\sin i] - (\cos i\sin 2\theta - \cos 2\theta\sin\varphi\sin i)P(d\Omega/d\varphi) + O(J^2) \tag{3}$$

Equation for P

$$(d/d\varphi)\{[P(d\Omega/d\varphi)](\cos i\sin^2\theta - \tfrac{1}{2}\sin 2\theta\sin\varphi\sin i)\} + (\sin\varphi\sin i(d\theta/d\varphi) + \tfrac{1}{2}\sin 2\theta\cos\varphi\sin i)P(d\Omega/d\varphi) + O(J^2) \tag{4}$$

In Ref. 1 no distinction was made between the variable $\varphi = (1 + J\varphi_1)\xi$ and ξ in the arguments of the trigonometric terms. It is clear that when $\xi = O(1/J)$ (i.e., for large times) such an approximation cannot be valid and will not be made here. In contrast, it is pointed out that trigonometric terms with the argument θ may be expanded since θ_1 will later be shown to be a bounded function of ξ.

Assuming the asymptotic expansions for the three variables u, θ, and P of Ref. 1, one obtains

$$u_0'' + u_0 = \mu/P_0^2 \tag{5}$$

$$\theta_0 = \pi/2 \tag{6}$$

$$P_0' = 0 \tag{7}$$

$$u_1'' + u_1 = -2\varphi_1 u_0 - 2u_0\omega\cos i + (\mu/P_0^2)\{2\varphi_1 - (2P_1/P_0) + R^2[u_0^2 + (u_0u_0'\sin 2\varphi)\sin^2 i - (3u_0^2\sin^2\varphi)\sin^2 i]\} \tag{8}$$

$$\theta_1'' + \theta_1 = (\mu R^2 u_0/P_0^2)\sin 2i\sin\varphi + 2\omega\sin\varphi\sin i \tag{9}$$

$$P_1' = -(\mu R^2 u_0/P_0)\sin 2\varphi\sin^2 i \tag{10}$$

where primes denote differentiation with respect to ξ, and $d\Omega/d\varphi = J\omega$ for some constant ω to be determined later.

If at time $t = 0$, the following general initial conditions apply

$$\begin{aligned} &r = r_0 && \theta = \frac{\pi}{2} && \varphi = \varphi_0 \\ &\frac{dr}{dt} = 0 && \frac{d\theta}{dt} = 0 && \frac{d\varphi}{dt} = \frac{V_0}{r_0} \end{aligned} \tag{11}$$

(Note that V_0, r_0, φ_0 are now measured in the moving frame.) The corresponding initial values when $\varphi = \varphi_0$ or $\xi = \xi_0 \equiv \varphi_0(1 - J\varphi_1)$ are

$$\begin{aligned} &u_0 = 1/r_0 && \theta_0 = \pi/2 && P_0 = r_0V_0 \\ &u_0' = 0 && \theta_0' = 0 && \\ &u_1 = 0 && \theta_1 = 0 && P_1 = 0 \\ &u_1' = 0 && \theta_1' = 0 && \end{aligned} \tag{12}$$

The solution $\theta_0 = \pi/2$ given in (6) follows from the limiting equation for θ as $J \to 0$, and the initial conditions $\theta_0 = \pi/2$ and $\theta_0' = 0$.

The solutions of Eqs. (5, 7, and 10) are

$$u_0 = K[1 + \eta\cos\psi] \tag{13}$$

$$P_0 = r_0V_0 \tag{14}$$

$$P_1 = (-R^2K^2P_0\sin^2 i/6)[(3 + 4\eta)\cos 2\varphi_0 - 3\eta\cos(2\varphi - \psi) - 3\cos 2\varphi - \eta\cos(2\varphi + \psi)] \tag{15}$$

where

$$K = \mu/P_0^2 = 1/r_0c^2$$

$$\psi = \xi - \xi_0$$

When the solution for u_0 given by (13) is substituted into (9) one obtains

$$\theta_1'' + \theta_1 = (K^2R^2\,\eta\sin 2i/2)[\sin(\varphi + \psi) + \sin(\varphi - \psi)] + \delta\sin\varphi \tag{16}$$

where

$$\delta = 2\,[K^2R^2\cos i + \omega]\sin i$$